

Predicted signatures of p -wave superfluid phases and Majorana zero modes of fermionic atoms in RF absorption

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We study the superfluid phases of quasi-2D atomic Fermi gases interacting via a p -wave Feshbach resonance. We calculate the absorption spectra of these phases under a hyperfine transition, for both non-rotating and rotating superfluids. We show that one can identify the different phases of the p -wave superfluid from the absorption spectrum. The absorption spectrum shows clear signatures of the existence of Majorana zero modes at the cores of vortices of the weakly-pairing $p_x + ip_y$ phase.

The observation of p -wave Feshbach resonances in cold fermion gases[1] opens up the possibility of the experimental realisation of unconventional superfluid states with non-zero pairing angular momentum[2, 3, 4]. It has recently been predicted[3, 4] that a one-component fermion gas interacting via a p -wave Feshbach resonance shows a series of p -wave superfluid phases, including a p_x phase, and both strong- and weak- pairing phases[6] of $p_x \pm ip_y$ symmetry. The strong- and weak- pairing phases differ in the nature of their excitation spectra, and are separated by a topological phase transition[6, 7].

Among these, the weak-pairing $p_x \pm ip_y$ phase has recently received considerable theoretical attention. The vortex excitations of this phase are predicted to hold gapless Majorana fermions on their cores, which should give rise to *non-abelian* exchange statistics in a quasi 2D geometry[5, 6]. Such excitations may have their uses in topologically-protected quantum computation [3, 8, 9]. However, to our knowledge, there as yet are no experimental observations of these very interesting physical properties in any realisation of a $p_x \pm ip_y$ superfluid. Developing methods to experimentally probe the properties of p -wave superfluid phases in cold atom systems is then of high and timely interest.

In this Letter we show that features in the RF absorption spectrum of a fermionic gas provide unambiguous signatures of the different p -wave superfluid phases. We show that the weak-pairing phases can be identified from the strong-pairing phases in the absorption spectrum of the superfluid at rest. Most significantly, we focus on the $p_x + ip_y$ phase and show that for a *rotating* fluid at weak-pairing, the absorption spectrum has clear and striking signatures of the zero energy Majorana fermion modes on the cores of the vortices. This spectrum is therefore a direct probe of the physics underlying the proposed non-abelian exchange statistics of this phase. (These experimental signatures do not require manipulation of positions of vortices or the use of other localised probes[9].) Furthermore, we show that the absorption spectrum for the rotated $p_x + ip_y$ weak-pairing phase depends on the *sense* of the rotation. This dependence is a direct indication of the time-reversal symmetry breaking in this

phase.

We consider a gas of identical fermionic atoms in an internal state that we denote $|\downarrow\rangle$, interacting via a p -wave Feshbach resonance. The phase diagram of the system has the same qualitative form in wide [4] and narrow [3] resonance limits. Depending on temperature, and on the detuning and anisotropic splitting of the Feshbach resonance, there appear superfluid phases with p_x and $p_x + ip_y$ symmetries. For $p_x + ip_y$, there is a transition between the weak-pairing phase (chemical potential $\mu > 0$) and the strong-pairing phase ($\mu < 0$) as the resonance is swept from positive to negative detuning[3]. We propose to probe the properties of these phases by measuring the absorption spectrum, under excitation of an atom in state $|\downarrow\rangle$ to a new internal state $|\uparrow\rangle$ via the perturbation

$$H_{\text{pert}} = \eta \int d^2r e^{i\mathbf{q} \cdot \mathbf{r} - i\omega t} \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) + \text{h.c.}, \quad (1)$$

where η sets the Rabi frequency of the transition. We consider $|\downarrow\rangle \rightarrow |\uparrow\rangle$ to be a hyperfine transition, driven by RF absorption[10, 11] or a two-photon transition[9]; measurement of the population of excited (or remaining) atoms is achieved by separate absorption on an electronic transition. We denote the energy splitting between the states $|\downarrow\rangle$ and $|\uparrow\rangle$ by E_g ; for a hyperfine transition, $E_g \sim 10^9 \text{ Hz}$, and we set the net photon momentum transfer to zero, $\mathbf{q} \rightarrow 0$. The $|\uparrow\rangle$ -atoms will not participate in the resonant p -wave interaction with the $|\downarrow\rangle$ -atoms, so for the most part we shall consider the $|\uparrow\rangle$ -atoms to be free. However, we shall discuss the effects of s -wave interspecies interactions. We shall focus on a uniform gas in a quasi-two-dimensional geometry, that is with confinement applied in the z -direction such that the associated confinement energy is large compared to the Fermi energy. This simplifies the analysis, allowing analytical calculations of the absorption features. The three dimensional case is left for a future work.

When the interaction between atoms may be neglected (far from the Feshbach resonance) and they form a simple Fermi gas, a transition of an atom from $|\downarrow\rangle$ to $|\uparrow\rangle$ requires an energy of E_g , leading to a delta-function peak

of the absorption at $\omega = E_g$. (We choose units for which $\hbar = 1$.) In contrast, we find that in the superfluid phases (close to the resonance) the absorption differs strongly for weak- and strong-paired superfluids. In the weak-pairing phase the energy E_g becomes a threshold energy for absorption, with non-zero weight at $\omega = E_g$ and a continuous spectrum above this (see Eq. (4)). In the strong pairing phase the threshold is shifted to $E_g + 2|\mu|$, with weight that is zero at $E_g + 2|\mu|$ and grows linearly with energy. The discontinuity in the absorption spectrum in the weak-pairing phase unambiguously identifies it from strong pairing.

In the weak-pairing $p_x \pm ip_y$ phase, the introduction of vortices by rotation leads to the creation of a set of gapless modes on the vortices. In the presence of a vortex lattice we find that *a series of equally spaced sub-gap peaks will be present in the absorption spectrum*, see Fig. [1]. These peaks are a result of transitions from the zero energy core states of \downarrow -atoms to various states of \uparrow -atoms, and have a weight that is linearly proportional to the number of vortices N_V . The peaks start at $\sim E_g - \mu$, and their weight decays monotonically in energy, becoming small well before the onset of the continuum at $E_g - \mu + \Delta$ (Δ is the superfluid gap). For a dense vortex lattice, in which there is tunneling between the Majorana modes, the peaks broaden to reveal a lineshape showing a van-Hove singularity, see inset of Fig. [1]. For the same weak-pairing phase, with vortices of the *opposite* sense of rotation, we find that the absorption in the Majorana modes again gives rise to a set of narrow absorption peaks. The peaks still start at $\sim E_g - \mu$, but now their weight rises linearly in energy, passing through a maximum well before the continuum. As we shall discuss below, the difference in spectra for the two senses of rotation is a signature of the fact that $p_x + ip_y$ breaks time-reversal symmetry.

We now provide the details of the calculations. The rate of excitations between the two hyperfine states is governed by the Golden Rule

$$\Gamma[\omega] = 2\pi|\eta|^2 \sum_{ab} |M_{ab}|^2 \delta(E_{\downarrow,a} + E_{\uparrow,b} - \omega), \quad (2)$$

where $E_{\downarrow,a}$ ($E_{\uparrow,b}$) is the energy required to produce an \downarrow -excitation (\uparrow -excitation), and M_{ab} is the dimensionless matrix element (measured in units of η) for producing excitations with quantum numbers a and b respectively. In the following we shall calculate $\Gamma[\omega]$, first for a system at rest, then for a rotating system.

We describe the superfluid phases within the Bogoliubov-de-Gennes (BdG) approach. For a uniform system, the Hamiltonian is given by

$$H = H_{0,\downarrow} + H_{\text{pair},\downarrow} + H_{0,\uparrow} + E_g \left(\frac{N_\uparrow - N_\downarrow}{2} \right), \quad (3)$$

where $H_{0,s} = \int d^2k \psi_s^\dagger(\mathbf{k}) \left(\frac{\mathbf{k}^2}{2m} - \mu \right) \psi_s(\mathbf{k})$, $s = \uparrow, \downarrow$,

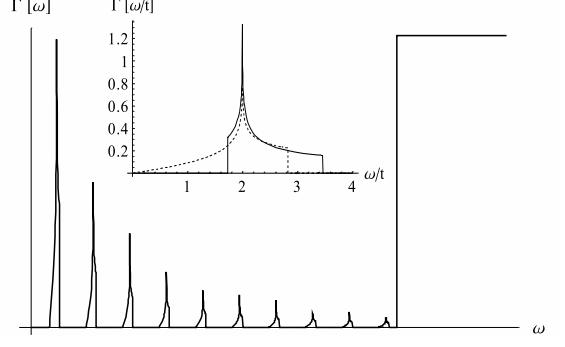


FIG. 1: Peak distribution associated with the Majorana modes on the vortices of the weak-pairing $p_x + ip_y$ phase. The first peak will be at $E \simeq E_g - \mu$, and other peaks will follow at spacings ω_c apart. Continuous absorption will be measured starting at $E \simeq E_g$. For the opposite sense of rotation, the weight of the peaks starts linearly and passes through a maximum. The inset shows the shape of a single peak for a square (dashed line) and triangular (solid line) lattice, showing a van-Hove singularity at $E = 2t$ (Γ is measured in units of $2\pi|\eta|^2 N_V \Theta_0/t$).

$H_{\text{pair},\downarrow} = \int d^2k \psi_\downarrow^\dagger(\mathbf{k}) v_\Delta(k_x + ik_y) \psi_\downarrow(-\mathbf{k}) + \text{h.c.}$. The values of μ and v_Δ may be obtained from the mean-field calculations of Refs. 3, 4. The Hamiltonian for \downarrow -atoms is diagonal in the Fock space of $\alpha_{\mathbf{k}} = u_{\mathbf{k}} \psi_{\downarrow,\mathbf{k}} + v_{\mathbf{k}} \psi_{\downarrow,-\mathbf{k}}^\dagger$ and $\alpha_{\mathbf{k}}^\dagger = u_{\mathbf{k}}^* \psi_{\downarrow,\mathbf{k}}^\dagger + v_{\mathbf{k}}^* \psi_{\downarrow,-\mathbf{k}}$ where $u_{\mathbf{k}}, v_{\mathbf{k}}$ are solutions of the BdG equation, $|u_{\mathbf{k}}|^2 = \frac{1}{2} \left[1 + \left(\frac{\mathbf{k}^2}{2m} - \mu \right) / E_{\downarrow,\mathbf{k}} \right]$ and $|v_{\mathbf{k}}|^2 = \frac{1}{2} \left[1 - \left(\frac{\mathbf{k}^2}{2m} - \mu \right) / E_{\downarrow,\mathbf{k}} \right]$, and $E_{\downarrow,\mathbf{k}} = \sqrt{\left(\frac{\mathbf{k}^2}{2m} - \mu \right)^2 + v_\Delta^2 \mathbf{k}^2}$ [3, 6]. Treating the excited \uparrow -atoms as free, $E_{\uparrow,\mathbf{k}} = \mathbf{k}^2/2m - \mu$, a summation over \mathbf{k} leads to the absorption spectrum [19]

$$\Gamma[\omega] = |\eta|^2 \frac{S}{v_\Delta^2} \frac{\delta\omega + 2\mu}{\left[1 + \frac{\delta\omega}{mv_\Delta^2} \right]^2} \times \begin{cases} \theta(\delta\omega), & \mu > 0 \\ \theta(\delta\omega - 2|\mu|), & \mu < 0, \end{cases} \quad (4)$$

where S is the area of the sample and $\delta\omega \equiv \omega - E_g$. Non-zero interactions of the excited \uparrow -atom with the \downarrow -atoms lead to a (mean-field) shift of both of these spectra [20].

We now turn to discuss the system under rotation, in a regime where a large vortex lattice is formed, $N_V \gg 1$. (The rotation frequency Ω is then close to the trapping frequency ω_T .) We first examine a superfluid in the weak pairing $p_x + ip_y$ phase, and concentrate on the regime $\Omega \ll \mu \ll mv_\Delta^2$.

Projected to the low-energy sector of the Majorana modes, the Hamiltonian for the \downarrow -atoms becomes a tight-binding Hamiltonian of the form [12]

$$H_\downarrow = H_{0,\downarrow} + H_{\text{pair},\downarrow} = it \sum_{\langle ij \rangle} s_{ij} \gamma_{i\downarrow} \gamma_{j\downarrow}, \quad (5)$$

where $s_{ij} = \pm$, in a way that corresponds to half a flux quantum per plaquette for a square lattice, and a quar-

ter of a flux quantum per plaquette for a triangular lattice. Here γ_i represents a Majorana fermion localized at \mathbf{R}_i . The parameter t describes the tunneling between the Majorana modes of nearby vortices; this tunneling is assumed small, $t \ll \mu$, which is valid if the separation between vortices, $\ell \equiv \sqrt{1/(2m\Omega)}$, is large compared to the spatial extent of the Majorana mode, r_0 . The Majorana wave function $g(\mathbf{r})$ has an oscillating contribution within the vortex core, and a decaying part outside the core. When $\mu \ll mv_\Delta^2$, the size of the core can be estimated as $r_{\text{core}} \sim \frac{v_F}{mv_\Delta^2} \log^{-1/2} \frac{v_F}{v_\Delta}$, with v_F being the Fermi velocity of the underlying fermionic gas, while the decay length outside the core is $r_0 = v_\Delta/\mu$. In the limit of small μ we have $r_0 \gg r_{\text{core}}$, such that we may approximate $g(\mathbf{r}) \approx \frac{e^{-r/r_0}}{\sqrt{2\pi r_0 r}}$, with r being the distance from the center of the vortex [13]. The analysis of the Hamiltonian (5) is presented in [12, 14]. For a square and triangular lattice the spectra are $E_{\square,\mathbf{k}} = 2t\sqrt{\sin^2(ak_x) + \sin^2(ak_y)}$ and $E_{\Delta,\mathbf{k}} = \sqrt{2}t\sqrt{3 + \cos(2ak_x) - 2\cos(ak_x)\cos(\sqrt{3}ak_y)}$ respectively[12].

The spectrum of the \uparrow -atoms in the rotating frame depends on the strength of the s -wave interspecies interactions. Since this depends sensitively on the atomic species used, we shall consider two limiting cases: For *vanishing interspecies interactions*, the \uparrow -atoms arrange into Landau levels due to rotation, with the cyclotron frequency $\omega_c = 2\Omega \sim 2\pi \times 10^2$ Hz. (The confinement frequency for \uparrow - and \downarrow -atoms is assumed the same.) For *strong interspecies repulsion*, the low energy states of the \uparrow -atoms are localized at the cores of vortices where the density of \downarrow -atoms is low, and can be described by a tight-binding Hamiltonian. The rotation then translates to a magnetic flux threading the lattice plaquettes (the Azbel-Hofstadter problem [15]).

Starting with the case of zero interspecies interactions ($V_\uparrow = 0$), the Hamiltonian $H_{0,\uparrow}$ can be diagonalized by expanding $\psi_\uparrow(\mathbf{r}) = \sum_{np} \phi_{np}(\mathbf{r}) c_{np}$, where $\phi_{np}(\mathbf{r})$ are wavefunctions for the n 'th Landau level, and p is some quantum number related the degeneracy of the Landau level. The rate of energy absorption is determined by

$$|M_{\alpha\mathbf{k},np}|^2 = \frac{v}{N_V} \sum_{ij} \lambda_{z(j),\mathbf{k}}^{\alpha*} \lambda_{z(i),\mathbf{k}}^{\alpha} e^{i\mathbf{k}\cdot(\mathbf{R}_i-\mathbf{R}_j)} \int d^2\mathbf{r} d^2\mathbf{r}' e^{\frac{i}{2}[\sum_{m \neq i} \arg(\mathbf{r}-\mathbf{R}_m) - \sum_{m \neq j} \arg(\mathbf{r}'-\mathbf{R}_m)]} g_i(\mathbf{r}) g_j(\mathbf{r}') \phi_{np}^*(\mathbf{r}') \phi_{np}(\mathbf{r}), \quad (6)$$

where $g_i(\mathbf{r}) = g(\mathbf{r} - \mathbf{R}_i)$. The functions $\lambda_{z(i),\mathbf{k}}^{\alpha}$ are listed in Ref.12, $z(i) = 1, \dots, v$ numbers the vortices contained in a unit cell, and $\alpha = \pm 1, \dots, \pm v/2$ is a band index. Since p enumerates degenerate states, we can now sum over it using the addition theorem for Landau levels [16]. We keep only the leading term $\mathbf{R}_i = \mathbf{R}_j$. Finally we expand the phases in Eq. (6) to first order in $\mathbf{r} - \mathbf{r}'$, and obtain $\sum_p |M_{\alpha\mathbf{k},np}|^2 = \frac{r_0^2}{\ell^2} \left[1 + 2n \left(\frac{r_0}{\ell} \right)^2 \right]^{-2}$. Thus the integrated absorption into the n 'th Landau level falls monotonically with increasing n . Plugging the matrix elements into Eq. (2), we arrive at the result

$$\Gamma[\omega] = 2\pi \frac{|\eta|^2}{t} N_V \sum_n \Theta_n F \left[\frac{|\omega| - \omega_n}{t} \right], \quad (7)$$

where $\Theta_n = \left(\frac{r_0}{\ell} \right)^2 \left[1 + 2n \left(\frac{r_0}{\ell} \right)^2 \right]^{-2}$, $\omega_n = E_g - \mu + \omega_c(n + 1/2)$, and F is a dimensionless function proportional to the density of states of the band formed by the Majorana operators, which is nonzero over a range of order unity. It is explicitly calculated below.

Using the results above, assuming $t \ll \omega_c$, we can find the integrated absorption over a single Landau level

$$\int_{-\frac{\omega_c}{2}}^{\frac{\omega_c}{2}} d(\omega - \omega_n) \Gamma = \pi |\eta|^2 N_V \Theta_n. \quad (8)$$

The integrated absorption over all frequencies for excitations from the zero modes can be found using the completeness of the Landau level functions $\sum_{np} |M_{\alpha\mathbf{k},np}|^2 = \frac{1}{2}$, hence $\int d\omega \Gamma[\omega] = \pi |\eta|^2 N_V / 2$.

When interspecies interaction is strong, the \uparrow -atoms feel a periodic potential V_\uparrow due to the vortex lattice structure. Due to depletion of \downarrow -atoms at a vortex core, the potential V_\uparrow possesses there a minimum, and may hold several bound states $\phi_n(\mathbf{r})$ for \uparrow -atoms. Several tight-binding bands will be formed, labelled by n . The \uparrow -atoms feel the same flux per plaquette as the Majorana fermions do. Therefore, the Hamiltonian for a single band of tight-binding atoms in the rotating frame may be written as

$$H_{0,\uparrow} = E'_n \sum_i c_{n,i\uparrow}^\dagger c_{n,i\uparrow} + i t'_n \sum_{\langle ij \rangle} s_{ij} c_{n,i\uparrow}^\dagger c_{n,j\uparrow}, \quad (9)$$

where t'_n is the tunneling matrix element for the \uparrow -atoms, and E'_n is an on-site energy. Diagonalizing the Hamiltonian, one finds that the spectrum of the \uparrow -atoms has the same \mathbf{k} -dependence as that of the Majorana fermions. The resulting absorption is given by Eq. (7) with $\Theta_n = |\int d^2\mathbf{r} g(\mathbf{r}) \phi_n(\mathbf{r})|^2$, $\omega_n = E_g - \mu + E'_n$, $t \rightarrow t + t'_n$.

We now proceed to calculate the function F appearing in Eq. (7). This function is $F[E/t] \equiv t\rho[E]/N_V$, with $\rho(E)$ being the density of states of the Majorana

fermions. Starting with a square lattice, $F[x]$ is

$$F_{\square}[x] = \frac{8x}{(2\pi)^2} \int_C \frac{dy}{\sqrt{x^2 - y^2} \sqrt{4 - x^2 + y^2} \sqrt{4 - y^2}}. \quad (10)$$

For $0 < x < 2$, $C = [0, x]$, while for $2 < x < 2\sqrt{2}$ $C = [\sqrt{x^2 - 4}, 2]$. We note the diverging density of states at $E = 2t$, and the discontinuity at the top of the band $E = 2\sqrt{2}t$. For a triangular lattice, we again find a logarithmic divergence near $E = 2t$ and discontinuities in the response at the top and the bottom of the band, $E = 2\sqrt{3}t$ and $E = \sqrt{3}t$ respectively.

For the *opposite* sense of rotation, the vortex lattice consists of an array of vortices of opposite circulation. These vortices also have zero modes[17]. However, the Majorana modes now have a non-trivial dependence around the vortex center, $g_i(\mathbf{r}) \propto e^{-i\arg(\mathbf{r} - \mathbf{R}_i)}$. (We again assumed $r_0 \gg r_{\text{core}}$.) As a consequence, the integrated weight of the absorption from the Majorana mode into the n 'th Landau level is proportional, for small n , to

$$\sum_p |M|^2 = \frac{9\pi}{64} \left(\frac{r_0}{\ell} \right)^4 \left(n + \frac{1}{2} \right). \quad (11)$$

The weight passes through a maximum at a frequency of about $\frac{\mu^2}{mv_\Delta^2}$ above the onset at $E_g - \mu$.

Thus we find that in the weak pairing $p_x + ip_y$ phase, the spectrum depends strongly on the sense of rotation. This dependence may be probed by preparing the system in the $p_x + ip_y$ phase and comparing spectra for rotations of the two senses. Alternatively, since what matters is the *relative* sense of rotation as compared to the internal angular momentum of the order parameter, the dependence could be tested by comparing the spectra of the $p_x + ip_y$ and $p_x - ip_y$ phases for the same sense of rotation. In a stationary fluid, the two phases are equally likely to be formed [21]. If for a given sense of rotation absorption spectra of two types are randomly observed for different realizations of nominally the same parameters, it would be a direct demonstration of time-reversal symmetry breaking. We note that the results may depend on how the state was formed (apply rotation and then cool, or cool and then apply rotation)[22].

The observation of absorption that originates from Majorana modes would also distinguish the time-reversal symmetric p_x phase from the phase of $p_x + i\beta p_y$, with a real $\beta \neq 0$, where time-reversal symmetry is broken. The latter may appear for an anisotropic Feshbach resonance[3, 4]. The distinction arises, as we now show, from the presence of Majorana modes for all $\beta \neq 0$, and their absence for $\beta = 0$. Solutions to the BdG equations come in pairs of $\pm E$. When a vortex carries a localized single zero energy state, as it does for $\beta = 1$, the only ways the energy of that state may shift due to a change of β are through mixing with other zero energy states (on other vortices, or on the sample's edge) or when the bulk

gap closes. When vortices are far away from one another and from the edge, the zero energy state exists as long as the gap does not close, i.e., as long as $\beta \neq 0$ [18][23].

To conclude, our results show that the absorption spectrum can be used to distinguish the various p -wave superfluid phases of an atomic Fermi gas, and to probe the breaking of time reversal symmetry. Furthermore, for a rotating superfluid in the weak pairing $p_x + ip_y$ phase, the absorption shows a series of sub-gap peaks, of unique shape and distribution, which are the direct consequence of the zero-energy Majorana modes on the vortices. The most favourable parameters for the observation of these sub-gap peaks are $\mu \ll mv_\Delta^2$ (to avoid finite energy bound states in the vortex core), and temperature $T \ll \mu$ (to avoid thermally excited quasi-particles which may wash out the absorption peaks). A practical way to observe the peaks would be by comparing measurements of the absorption for increasing rotation frequency. This will vary both the strength of the peaks (due to the increase of N_V) and their spacing (due to the increase of ω_c).

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- [19] The unphysical logarithmic divergence of the integrated intensity in (4) is a reflection of the need to introduce a UV cut-off[4] to regularise the theory for $\Delta_k \propto k_x + ik_y$.
- [20] There may also be a renormalisation of the effective mass of the excited particle, and a damping for $k \neq 0$ modes.
- [21] There is also the possibility of formation of an inhomogeneous state with domains of the two phases.
- [22] In a rotating system time-reversal symmetry is explicitly broken, so, in a “rotation cooled” experiment, one phase may form in preference over the other.
- [23] We note that the anisotropic order parameter of the $p_x + i\beta p_y$ phase will choose a preferred direction for tunneling between vortices.